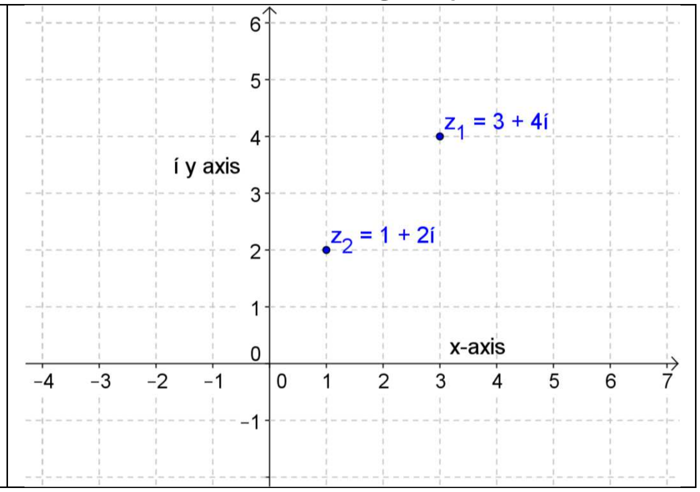
Determine Trigonometric Forms of Complex Numbers

You may recall the imaginary number $i = \sqrt{-1}$, as one of the solutions to the equation $x^2 = -1$. Observe that $i^2 = \left(\sqrt{-1}\right)^2 = -1$. Numbers like $4i$ and $2i$ are called **imaginary** numbers.

A complex number, z, is of the form $z = a + bi$ and can be graphed in the complex plane with the coordinate (a, b). In the graph to the right we have the complex numbers $z_1 = 3 + 4i$ corresponding to the coordinates $(3,4)$, and $z_2 = 1 + 2i$ corresponding to the coordinates $(1,2)$.



You can perform the operations addition, subtraction, multiplication and division on complex numbers as illustrated below.

Addition
To add, add the real components and the imaginary components.

$$z_1 + z_2 = (3 + 4i) + (1 + 2i)$$
$$= (3 + 1) + (2 + 4)i$$
$$= 4 + 6i$$

Subtraction
To subtract, subtract the real components and the imaginary components.

$$z_1 - z_2 = (3 + 4i) - (1 + 2i)$$
$$= (3 - 1) + (2 - 4)i$$
$$= 2 + 2i$$

Multiplication
To multiply use the usual method with binomial multiplication and the idea that $\left(\sqrt{-1}\right)^2 = -1$

$$z_1 z_2 = (3 + 4i)(1 + 2i)$$
$$= 3(1 + 2i) + 4i(1 + 2i)$$
$$= 3 + 6i + 4i + 8i^2$$
$$= 3 + 6i + 4i - 8 = -5 + 10i$$

Observe what happens when you multiply the complex number $z_1 = 3 + 4i$ by its **complex conjugate** $\overline{z_1} = 3 - 4i$

$$z_1 \overline{z_1} = (3 + 4i)(3 - 4i) = (9 + 16) = 25$$

You need the conjugate to divide as shown.

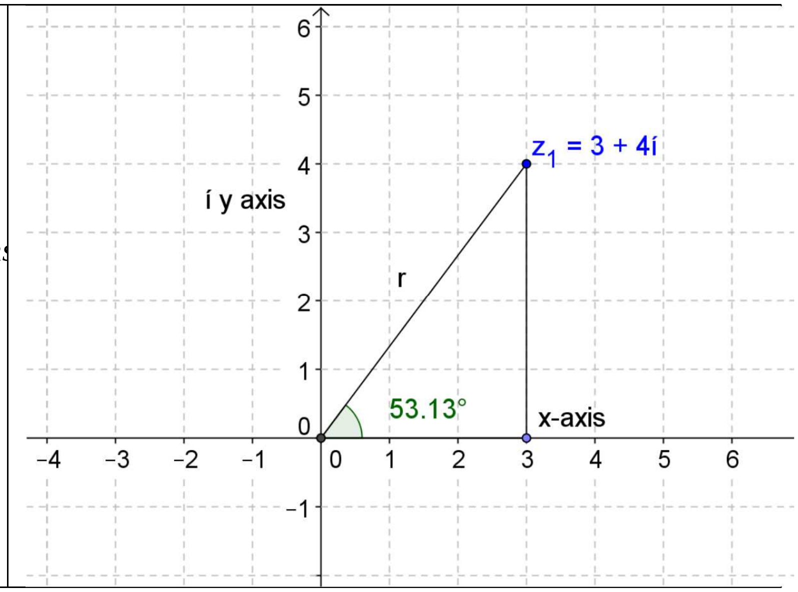
$$\frac{z_1}{z_2} = \frac{(3 + 4i)}{(1 + 2i)} = \frac{(3 + 4i)}{(1 + 2i)}\frac{(1 - 2i)}{(1 - 2i)}$$
$$= \frac{(11 - 2i)}{(5)} = \frac{11}{5} - \frac{2i}{5}$$

Like vectors the **direction** of $z_1 = 3 + 4i$ is indicated by going right 3 and up 4 or a slope, $m = \tan\theta = \frac{4}{3}$.

The direction is then the angle

$$\theta = \tan^{-1}\left(\frac{4}{3}\right) \approx 53.13^0 \approx 0.927 radians$$

And like vectors the **magnitude** is $r = |z_1| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5.$



You can also write complex numbers in trigonometric form using the graph on the right.

We have $\cos\theta = \frac{a}{r}$ and $\sin\theta = \frac{b}{r}$ so $a = r\cos\theta$ and $b = r\sin\theta$ so by substituting we have:

$$z = a + bi = r\cos + ri\sin\theta$$
$$= r(\cos\theta + i\sin\theta)$$

Recall from above that for $z_1 = 3 + 4i$

$$\theta = \tan^{-1}\left(\frac{4}{3}\right) \approx 53.13^0 \approx .927 radians$$ and

$r = |z_1| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5.$

So we have $z_1 = 3 + 4i$

$$= 5(\cos\ 0.927 + i\sin\ 0.927)$$